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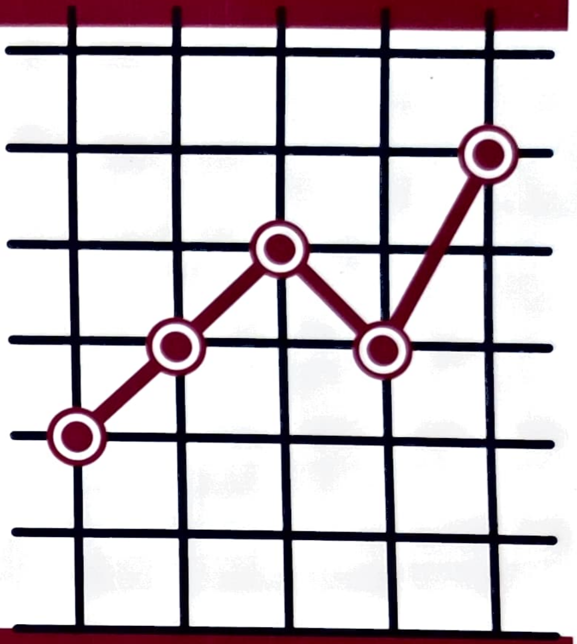


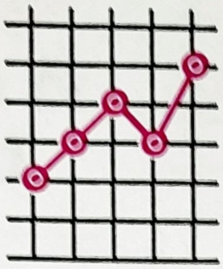
Skills in Mathematics for JEE Main & Advanced

Play with Graphs

With Sessionwise Theory & Exercises...

Amit M. Agarwal

Skills in
Mathematics for
**JEE Main &
Advanced**



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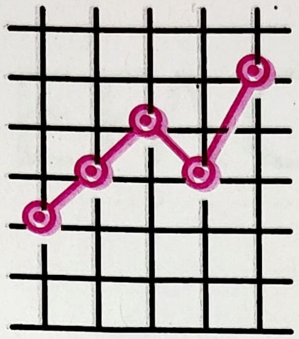

Play with Graphs

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ARIHANT PRAKASHAN (Series), MEERUT

PREFACE

It is a matter of great pleasure and pride for me to introduce to you this book "Play with Graphs". As a teacher, guiding the Engineering aspirants for over a decade now, I have always been in the lookout for right approach to understand various mathematical problems. I had always felt the need of a book that can develop and sharpen the ideas of the students within a very short span of time.

The book in your hands, aims to help you solve various mathematical problems by the use of graphs. Ways to draw different types of graphs are very easy and can be understood by even an average student. I feel glad to mention that the use of graphs in solving various mathematical problems has been well illustrated in this book.

I would like to take this opportunity to thank
M/s Arihant Prakashan for assigning this work to me.
It is their inspiration that has encouraged me to bring this book in this present form.

I would also like to thank Arihant DTP Unit for the nice laser typesetting.

Valuable suggestions from the students and teachers are always welcome, and these will find due places in the ensuing editions.

Amit M. Agarwal

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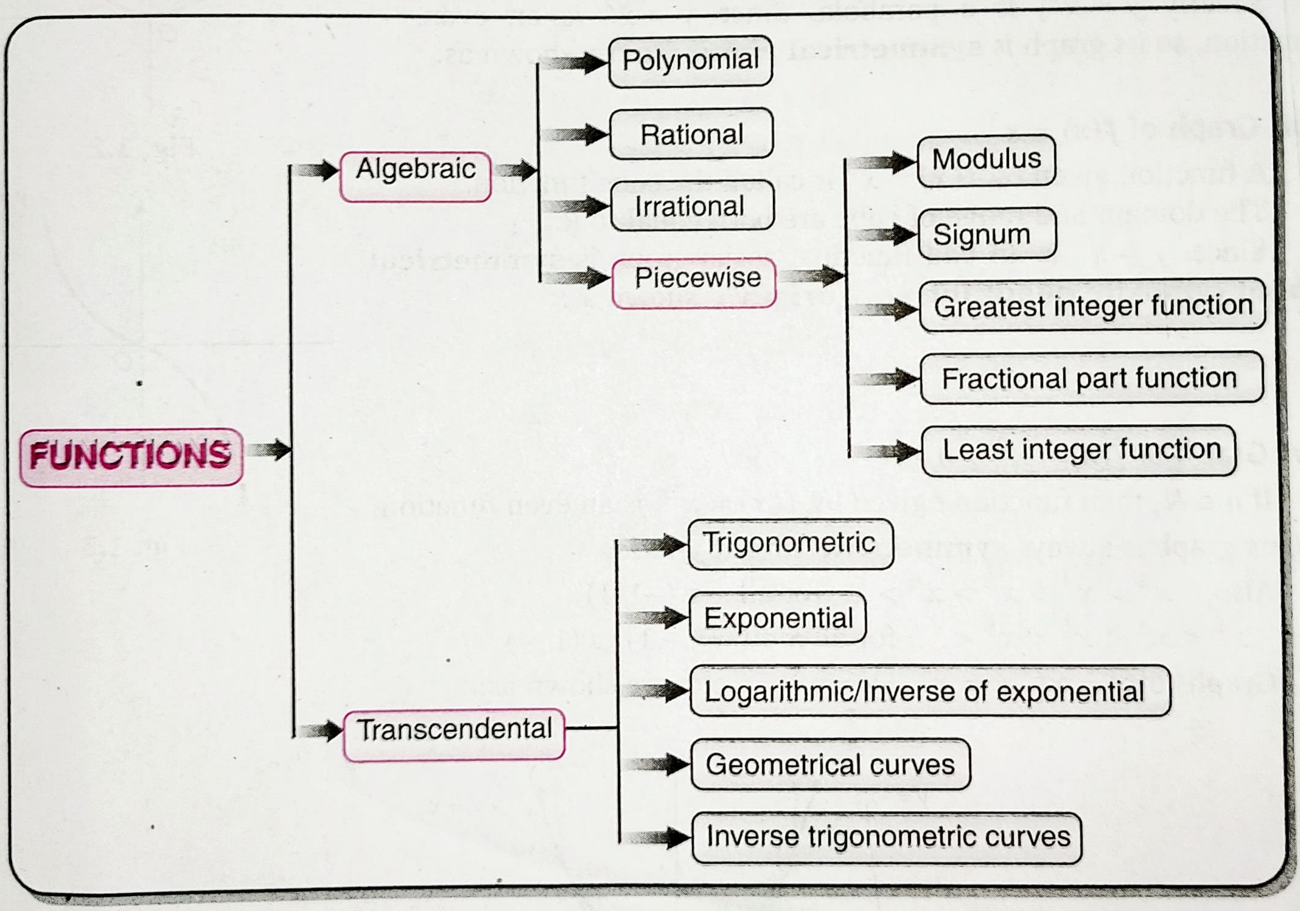
Chapter 1

INTRODUCTION OF GRAPHS

➡ **In this section, we shall revise some basic curves which are given as.**



1.1 ALGEBRAIC FUNCTIONS

1. Polynomial Function

A function of the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \ ;$$

where $n \in N$ and $ga_0, a_1, a_2, \dots, a_n \in R$.

Then, f is called a polynomial function. "$f(x)$ is also called polynomial in x".

Some of basic polynomial functions are

(i) Identity function/Graph of $f(x) = x$

A function f defined by $f(x) = x$ for all $x \in R$, is called the identity function.

Here, $y = x$ clearly represents a straight line passing through the origin and inclined at an angle of 45° with x-axis shown as:

The domain and range of identity functions are both equal to R.

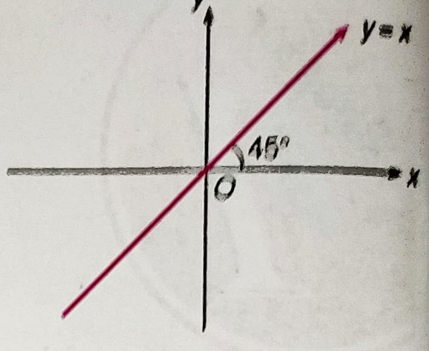


Fig. 1.1

(ii) Graph of $f(x) = x^2$

A function given by $f(x) = x^2$ is called the square function.

The domain of square function is R and its range is $R^+ \cup \{0\}$ or $[0, \infty)$

Clearly $y = x^2$, is a parabola. Since $y = x^2$ is an even function, so its graph is **symmetrical about y-axis**, shown as:

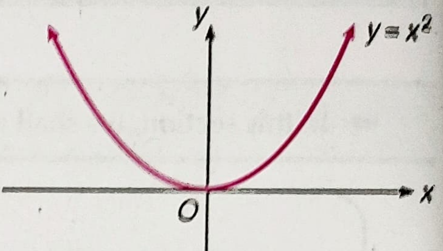


Fig. 1.2

(iii) Graph of $f(x) = x^3$

A function given by $f(x) = x^3$ is called the cube function.

The domain and range of cube are both equal to R.

Since, $y = x^3$ is an odd function, so its graph is **symmetrical about opposite quadrant**, *i.e.*, **"origin"**, shown as:

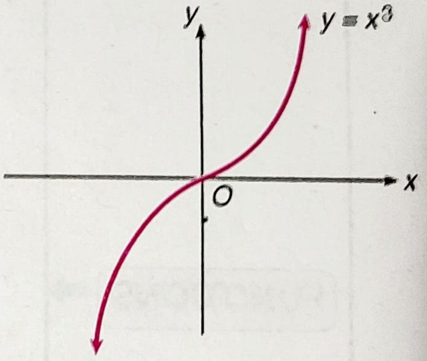


Fig. 1.3

(iv) Graph of $f(x) = x^{2n}$; $n \in N$

If $n \in N$, then function f given by $f(x) = x^{2n}$ is an even function. So, its graph is always **symmetrical about y-axis.**

Also, $x^2 > x^4 > x^6 > x^8 > \dots$ for all $x \in (-1, 1)$

and $x^2 < x^4 < x^6 < x^8 < \dots$ for all $x \in (-\infty, -1) \cup (1, \infty)$

Graphs of $y = x^2$, $y = x^4$, $y = x^6$,..., etc. are shown as:

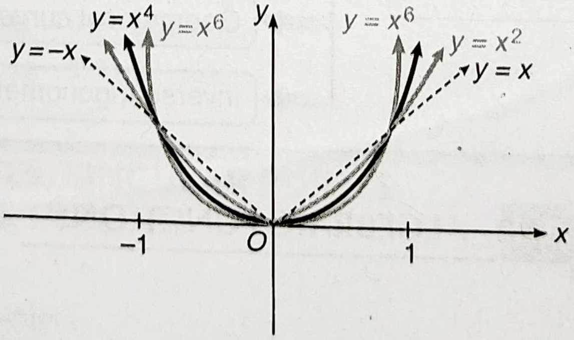


Fig. 1.4

(v) Graph of $f(x) = x^{2n-1}$; $n \in N$

If $n \in N$, then the function f given by $f(x) = x^{2n-1}$ is an odd function. So, its graph is **symmetrical about origin or opposite quadrants.**

Here, comparison of values of $x, x^3, x^5, \ldots$ for

$x \in (1, \infty)$	$x < x^3 < x^5 < \ldots$
$x \in (0, 1)$	$x > x^3 > x^5 > \ldots$
$x \in (-1, 0)$	$x < x^3 < x^5 < \ldots$
$x \in (-\infty, -1)$	$x > x^3 > x^5 > \ldots$

Graphs of $f(x) = x$, $f(x) = x^3$, $f(x) = x^5, \ldots$ are shown as in Fig. 1.5

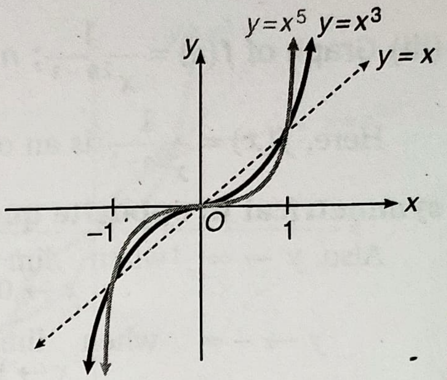


Fig. 1.5

2. Rational Expression

A function obtained by dividing a polynomial by another polynomial is called a rational function.

$$\Rightarrow \qquad f(x) = \frac{P(x)}{Q(x)}$$

$$\text{Domain} \in R - \{x \mid Q(x) = 0\}$$

i.e., domain $\in R$ except those points for which denominator $= 0$.

Graphs of some Simple Rational Functions

(i) Graph of $f(x) = \dfrac{1}{x}$

A function defined by $f(x) = \dfrac{1}{x}$ is called the reciprocal function or rectangular hyperbola, with coordinate axis as asymptotes. The domain and range of $f(x) = \dfrac{1}{x}$ is **$R - \{0\}$**.

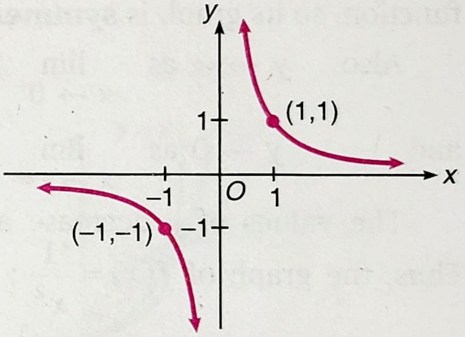


Fig. 1.6

Since, $f(x)$ is odd function, so its graph is **symmetrical about opposite quadrants.** Also, we observe

$$\lim_{x \to 0^+} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -\infty \,.$$

and $\qquad$ as $x \to \pm\infty \Rightarrow f(x) \to 0$

Thus, $f(x) = \dfrac{1}{x}$ could be shown as in Fig. 1.6.

(ii) Graph of $f(x) = \dfrac{1}{x^2}$

Here, $f(x) = \dfrac{1}{x^2}$ is an even function, so its graph is **symmetrical about y-axis.**

Domain of $f(x)$ is $R - \{0\}$ and range is $(0, \infty)$.

Also, as $y \to \infty$ as $\lim\limits_{x \to 0^+} f(x)$ or $\lim\limits_{x \to 0^-} f(x)$.

and $\qquad y \to 0$ as $\lim\limits_{x \to \pm\infty} f(x)$.

Thus, $f(x) = \dfrac{1}{x^2}$ could be shown as in Fig. 1.7.

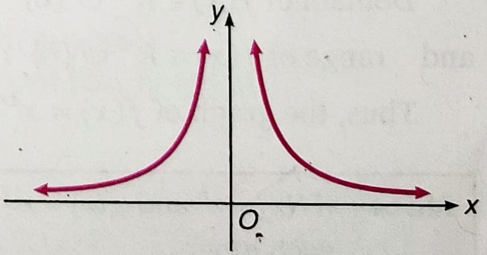


Fig. 1.7

(iii) Graph of $f(x) = \dfrac{1}{x^{2n-1}}$; $n \in N$

Here, $f(x) = \dfrac{1}{x^{2n-1}}$ is an odd function, so its graph is **symmetrical in opposite quadrants.**

Also, $y \to \infty$ when $\lim\limits_{x \to 0^+} f(x)$ and

$y \to -\infty$ when $\lim\limits_{x \to 0^-} f(x)$.

Thus, the graph for $f(x) = \dfrac{1}{x^3}$; $f(x) = \dfrac{1}{x^5}, \ldots,$ etc. will be similar to the graph of $f(x) = \dfrac{1}{x}$ **which has asymptotes as coordinate axes,** shown as in Fig. 1.8

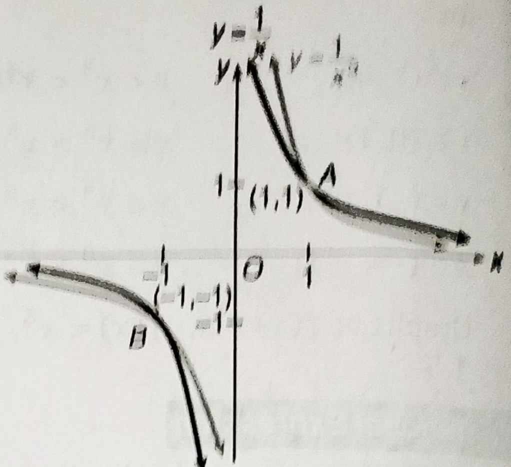


Fig. 1.8

(iv) Graph of $f(x) = \dfrac{1}{x^{2n}}$; $n \in N$

We observe that the function $f(x) = \dfrac{1}{x^{2n}}$ is an even function, so its graph is **symmetrical about y-axis.**

Also, $y \to \infty$ as $\lim\limits_{x \to 0^+} f(x)$ or $\lim\limits_{x \to 0^-} f(x)$

and $y \to 0$ as $\lim\limits_{x \to -\infty} f(x)$ or $\lim\limits_{x \to +\infty} f(x)$.

The values of y decrease as the values of x increase.

Thus, the graph of $f(x) = \dfrac{1}{x^2}$; $f(x) = \dfrac{1}{x^4}, \ldots,$ etc. will be similar as the graph of $f(x) = \dfrac{1}{x^2}$, which has **asymptotes as coordinate axis.** Shown as in Fig. 1.9.

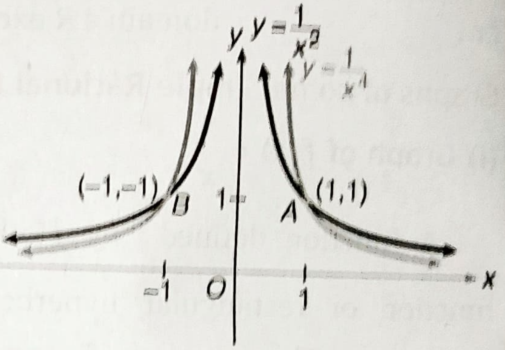


Fig. 1.9

3. Irrational Function

The algebraic function containing terms having non-integral rational powers of x are called irrational functions.

Graphs of Some Simple Irrational Functions

(i) Graph of $f(x) = x^{1/2}$

Here; $f(x) = \sqrt{x}$ is the portion of the parabola $y^2 = x$, which lies above x-axis.

Domain of $f(x) \in R^+ \cup \{0\}$ or $[0, \infty)$

and range of $f(x) \in R^+ \cup \{0\}$ or $[0, \infty)$.

Thus, the graph of $f(x) = x^{1/2}$ is shown as;

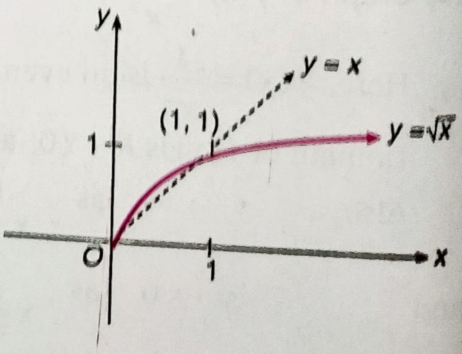


Fig. 1.10

> **Note** If $f(x) = x^n$ and $g(x) = x^{1/n}$, then $f(x)$ and $g(x)$ are inverse of each other.
> $\therefore$ $f(x) = x^n$ and $g(x) = x^{1/n}$ is the mirror image about $y = x$.

(ii) Graph of $f(x) = x^{1/3}$

As discussed above, if $g(x) = x^3$. Then $f(x) = x^{1/3}$ is image of $g(x)$ about $y = x$.

where domain $f(x) \in R$.

and range of $f(x) \in R$.

Thus, the graph of $f(x) = x^{1/3}$ is shown in Fig. 1.11;

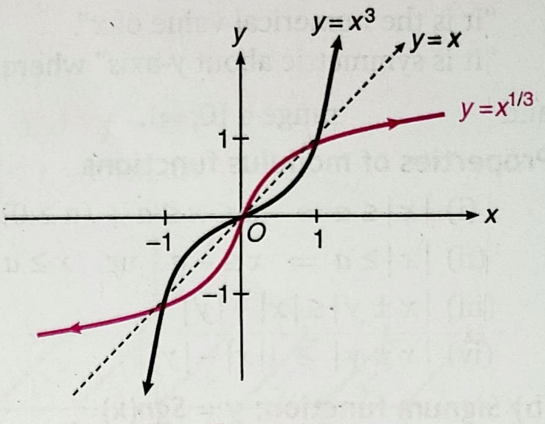


Fig. 1.11

(iii) Graph of $f(x) = x^{1/2n}$; $n \in N$

Here, $f(x) = x^{1/2n}$ is defined for all $x \in [0, \infty)$ and the values taken by $f(x)$ are positive.

So, domain and range of $f(x)$ are $[0, \infty)$.

Here, the graph of $f(x) = x^{1/2n}$ is the mirror image of the graph of $f(x) = x^{2n}$ about the line $y = x$, when $x \in [0, \infty)$.

Thus, $f(x) = x^{1/2}$, $f(x) = x^{1/4}$, ... are shown as;

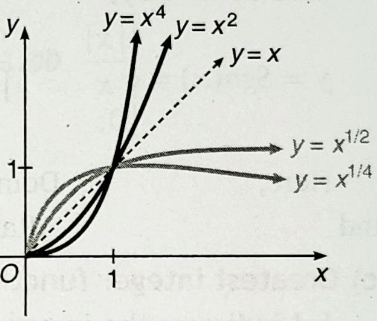


Fig. 1.12

(iv) Graph of $f(x) = x^{1/2n-1}$, when $n \in N$

Here, $f(x) = x^{1/2n-1}$ is defined for all $x \in R$. So, domain of $f(x) \in R$, and range of $f(x) \in R$. Also the graph of $f(x) = x^{1/2n-1}$ is the mirror image of the graph of $f(x) = x^{2n-1}$ about the line $y = x$ when $x \in R$.

Thus, $f(x) = x^{1/3}$, $f(x) = x^{1/5}$, ..., are shown as;

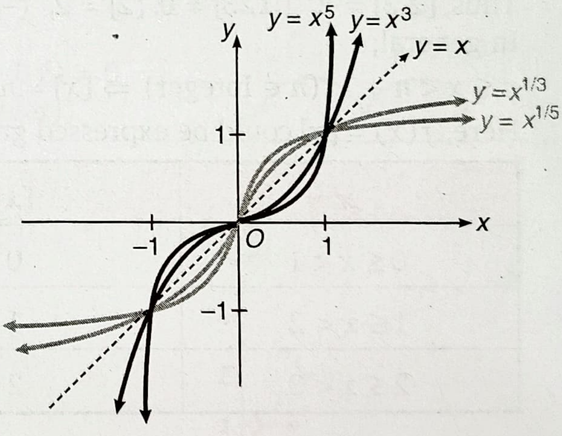


Fig. 1.13

Note We have discussed some of the simple curves for Polynomial, Rational and Irrational functions. Graphs of the some more difficult rational functions will be discussed in chapter 3. Such as;

$$y = \frac{x}{x+1}, \quad y = \frac{1}{x^2-1}, \quad y = \frac{x^2+x+1}{x^2-x+1}, \ldots .$$

4. Piecewise Functions

As discussed piecewise functions are:

(a) Absolute value function (or modulus function), (b) Signum function.
(c) Greatest integer function. (d) Fractional part function.
(e) Least integer function.

(a) Absolute value function (or modulus function)

$$y = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

"It is the numerical value of x".
"It is symmetric about y-axis" where domain $\in R$

and range $\in [0, \infty)$.

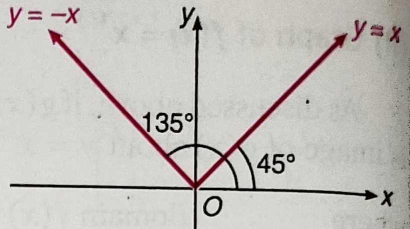


Fig. 1.14

Properties of modulus functions

(i) $|x| \le a \Rightarrow -a \le x \le a$; $(a \ge 0)$

(ii) $|x| \ge a \Rightarrow x \le -a$ or $x \ge a$; $(a \ge 0)$

(iii) $|x \pm y| \le |x| + |y|$

(iv) $|x \pm y| \ge ||x| - |y||$.

(b) Signum function; $y = Sgn(x)$

It is defined by;

$$y = Sgn(x) = \begin{cases} \dfrac{|x|}{x} \text{ or } \dfrac{x}{|x|}; & x \ne 0 \\ 0; & x = 0. \end{cases} = \begin{cases} +1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0. \end{cases}$$

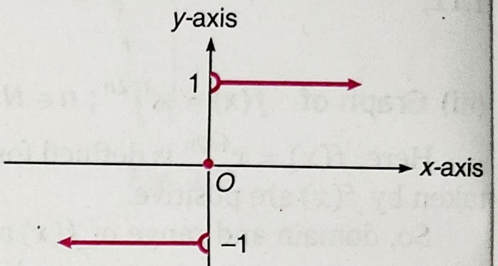


Fig. 1.15

Here, Domain of $f(x) \in R$.

and Range of $f(x) \in \{-1, 0, 1\}$.

(c) Greatest integer function

$[x]$ indicates the integral part of x which is nearest and smaller integer to x. It is also known as **floor of x**.

Thus, $[2.3] = 2$, $[0.23] = 0$, $[2] = 2$, $[-8.0725] = -9, \ldots$.

In general;

$n \le x < n + 1$ ($n \in$ Integer) $\Rightarrow [x] = n$.

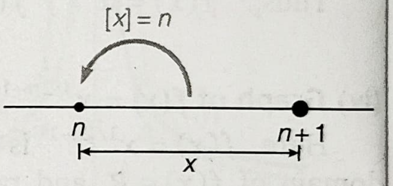


Fig. 1.16

Here, $f(x) = [x]$ could be expressed graphically as;

x	$[x]$
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2

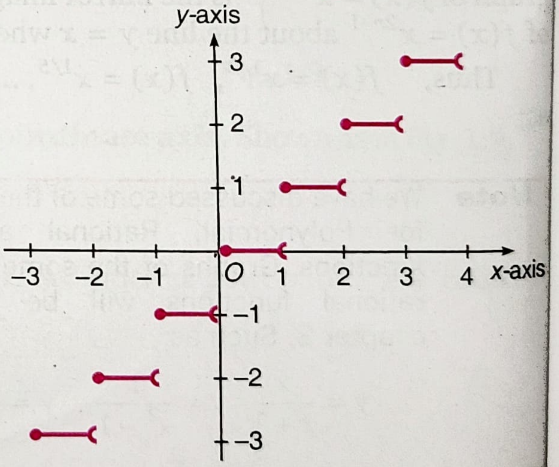


Fig. 1.17

Thus, $f(x) = [x]$ could be shown as;

Properties of greatest integer function

(i) $[x] = x$ holds, if x is integer.

(ii) $[x + I] = [x] + I$, if I is integer.

(iii) $[x + y] \ge [x] + [y]$.

(iv) If $[\phi(x)] \ge I$, then $\phi(x) \ge I$.

(v) If $[\phi(x)] \le I$, then $\phi(x) < I + 1$.

(vi) $[-x] = -[x]$, if $x \in$ integer.

(vii) $[-x] = -[x] - 1$, if $x \notin$ integer.

"It is also known as **stepwise function/floor of x.**"

(d) Fractional part of function

Here, {.} denotes the fractional part of x. Thus, in $y = \{x\}$.

$$x = [x] + \{x\} = I + f\ ; \text{ where } I = [x] \text{ and } f = \{x\}$$

$$\therefore \quad y = \{x\} = x - [x], \text{ where } 0 \le \{x\} < 1; \text{ shown as:}$$

x	$\{x\}$
$0 \le x < 1$	x
$1 \le x < 2$	$x - 1$
$2 \le x < 3$	$x - 2$
$-1 \le x < 0$	$x + 1$
$-2 \le x < -1$	$x + 2$

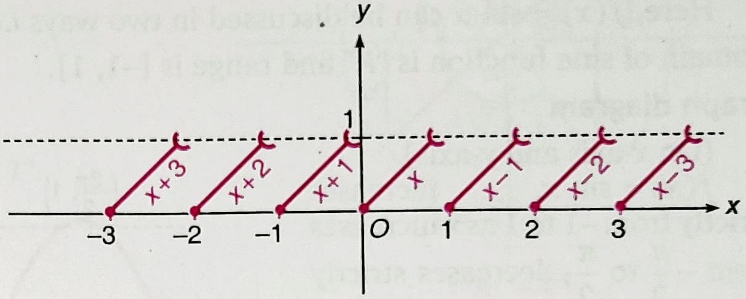


Fig. 1.18

Properties of fractional part of x

(i) $\{x\} = x$; if $0 \le x < 1$
(ii) $\{x\} = 0$; if $x \in$ integer.
(iii) $\{-x\} = 1 - \{x\}$; if $x \in$ integer.

(e) Least integer function

$$y = (x) = \lceil x \rceil,$$

(x) or $\lceil x \rceil$ indicates the integral part of x which is nearest and greatest integer to x.

It is known as ceiling of x.

Thus, $\lceil 2.3023 \rceil = 3$, $(0.23) = 1$, $(-8.0725) = -8$, $(-0.6) = 0$

In general, $n < x \le n + 1$ ($n \in$ integer))

i.e., $\lceil x \rceil$ or $(x) = n + 1$

shown as;

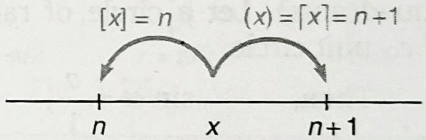


Fig. 1.19

Here, $f(x) = (x) = \lceil x \rceil$, can be expressed graphically as:

x	$\lceil x \rceil = (x)$
$-1 < x \le 0$	0
$0 < x \le 1$	1
$1 < x \le 2$	2
$-2 < x \le -1$	-1
$-3 < x \le -2$	-2

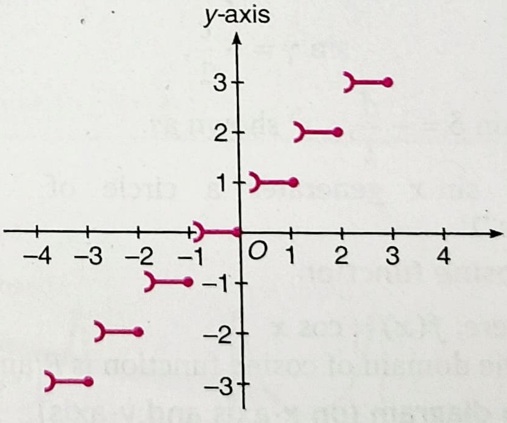


Fig. 1.20

Properties of least integer function

(i) $(x) = \lceil x \rceil = x$, if x is integer.
(ii) $(x + I) = \lceil x + I \rceil = (x) + I$; if $I \in$ integer.
(iii) Greatest integer converts $x = I + f$ to $[x] = I$ while $\lceil x \rceil$ converts to $(I + 1)$.

Note We shall discuss the curves:
$y = \{\sin x\}$, $y = \{x^3\}$, $y = \{\sin^{-1} (\sin x)\}$ $y = [\sin x]$, etc. in **chapter 2. (Curvature and Transformations).**